Spectral gap characterization of full type III factors

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Full factors

Let M be a von Neumann algebra (with separable predual). We equip the group $\mathrm{Aut}(M)$ with the following topology: a sequence $(\theta_n)_{n\in\mathbb{N}}$ converges to θ in $\mathrm{Aut}(M)$ if and only if

$$\lim_n \|\varphi \circ \theta_n - \varphi \circ \theta\| = 0$$

for every normal linear functionnal $\varphi \in M_*$. This turns $\mathrm{Aut}(M)$ into a Polish group.

Definition (Connes 1974)

A factor M is full when the subgroup of inner automorphisms $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$.

Remark: If M is a full factor, the quotient group $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ is a Polish group for the quotient topology.

Fullness and property Gamma

Definition (Murray & von Neumann)

A II_1 factor M has property (Γ) if there exists a *uniformly bounded* sequence $x_n \in M, \ n \in \mathbb{N}$ such that $\tau(x_n) = 0$ and $\lim_n \|ax_n - x_n a\|_2 = 0$ for all $a \in M$.

Theorem (Connes 1974)

A II_1 factor M is full if and only if it does not have property (Γ).

Ex 1: The hyperfinite II_1 factor $R = \bigotimes_{n\in\mathbb{N}} M_2(\mathbb{C})$ is not full.
Ex 2: The free group factor $\mathcal{L}(\mathbb{F}_2)$ is full. In fact, Murray & von Neumann proved the following inequality:

$$\forall x \in \mathcal{L}(\mathbb{F}_2), \ ||x - \tau(x)||_2 \leq 14(||u_a x - x u_a||_2 + ||u_b x - x u_b||_2)$$

where a and b are the two generators of $\mathbb{F}_2$.

In 1976, Connes published his celebrated work on the classification of injective factors. As a key step in his proof of injectivity $\Rightarrow$ hyperfiniteness, he proved the following remakable characterization of full factors:

Theorem (Connes 1976)

Let M be a II_1 factor. Then M is full if and only if there exists $u_1, \ldots, u_n \in \mathcal{U}(M)$ and a constant $\kappa > 0$ such that

$$\forall x \in M, \quad ||x - \tau(x)||_2 \leq \kappa \sum_{k=1}^{n} ||u_k x - x u_k||_2$$

This property means that the adjoint representation

$$\mathrm{Ad} : u \in \mathcal{U}(M) \mapsto uJuJ \in \mathrm{B}(\mathrm{L}^2(M))$$

has spectral gap.

Applications

Theorem (Connes 1976)

Let M and N be two II_1 factors. Then $M \mathbin{\overline{\otimes}} N$ is full if and only if both M and N are full.

Theorem (Popa 2006)

Let M and N be two full II_1 factors. Suppose that $M \mathbin{\overline{\otimes}} R \simeq N \mathbin{\overline{\otimes}} R$ where R is the hyperfinite II_1 factor. Then $M \simeq N^t$ for some $t > 0$.

Theorem (Jones 1981)

Let M be a full II_1 factor. Suppose that $\sigma : \Gamma \to \mathrm{Aut}(M)$ is an outer action of a discrete group Γ such that the image of $\sigma(\Gamma)$ in $\mathrm{Out}(M)$ is discrete. Then $M \rtimes_\sigma \Gamma$ is also a full factor.

Full type III factors

Question: can we generalize this results to arbitrary full factors (possibly of type III)?

Let M be a von Neumann algebra. There is a canonical Hilbert space $L^2(M)$ associated to M. Every element $a \in M$ acts on $L^2(M)$ on the left $\xi \mapsto a\xi$ and on the right $\xi \mapsto \xi a$.

Theorem (Connes 1974)

Let M be a factor. Then M is full if and only if for every uniformly bounded sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $\|x_n\xi - \xi x_n\|_2 \to 0$ for all $\xi \in L^2(M)$, there exists $\lambda_n \in \mathbb{C}$ such that $x_n - \lambda_n \to 0$ strongly.

Remark: if M is of type III, there is no relation between the fullness of M and the spectral gap property for the adjoint representation $\mathrm{Ad} : \mathcal{U}(M) \to \mathrm{B}(\mathrm{L}^2(M))$.

So how do we generalize Connes' characterization of full factors?

Theorem (M. 2016)

Let M be a full type III factor. There exist a faithful normal state $\varphi \in M_*$, a family $\xi_1, \dots, \xi_n \in L^2(M)$ and a constant $\kappa > 0$ such that

$$\forall x \in M, \quad ||x - \varphi(x)||_\varphi \leq \kappa \sum_{k=1}^{n} ||x\xi_k - \xi_k x||_2$$

Lemma

It is enough to show that there exists $\kappa > 0$ and $\xi_1, ..., \xi_n \in \mathrm{L}^2(M)$ such that

$$\varphi(p)(1 - \varphi(p)) \leq \kappa \sum_{k=1}^{n} \|p\xi_k - \xi_k p\|_2^2$$

for every **projection** *$p \in M$.*

Application

Today, we focus on the following application. It is a generalization of Jones' theorem to arbitrary factors.

Theorem (M. 2016)

Let M be a full factor (possibly of type III). Let $\sigma : \Gamma \to \mathrm{Aut}(M)$ an outer action of a discrete group Γ such that the image of $\sigma(\Gamma)$ in $\mathrm{Out}(M)$ is discrete. Then $M \rtimes_\sigma \Gamma$ is discrete.

Lemma

Let $\mathcal{V}$ be neighborhood of the identity in $\mathrm{Out}(M)$. Then we can find $\xi_1, \dots, \xi_n \in \mathrm{L}^2(M)$ such that

$$\|x\|_\varphi \leq \kappa \sum_{k=1}^{n} \|\theta(x)\xi_k - \xi_k x\|_2$$

for all $x \in M$ and all $\theta \in \mathrm{Aut}(M) \setminus \mathcal{V}$.

Modular theory

Let M be a von Neumann algebra with a faithful normal state φ.

Tomita-Takesaki's theory: the state φ produces a one-parameter group of automorphisms $t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(M)$ called the modular flow of φ. This modular flow is trivial iff φ is a trace.

Connes's cocycle theorem: if ψ is an other state, there exists a cocyle $u_t \in \mathcal{U}(M)$ such that $\sigma_t^\psi = \mathrm{Ad}(u_t) \circ \sigma_t^\varphi$.

Two consequences:

- The one-parameter group $\delta : t \in \mathbb{R} \mapsto [\sigma_t^\varphi] \in \mathrm{Out}(M)$ does not depend on φ.
- The crossed product $c(M) = M \times_{\sigma^\varphi} \mathbb{R}$ does not depend on φ up to canonical isomorphism. It is called the core of M.

Type III_1 factors with full core

Let M be a factor of type $\mathrm{III}_1 \Leftrightarrow c(M)$ is factor.
Question: when is $c(M)$ a full factor?

Theorem (M. 2016)

Let M be a type III_1 factor. Then $c(M)$ is full if and only if M is full and $\delta : \mathbb{R} \to \mathrm{Out}(M)$ is a homeomorphism on its range.

- The "only if" part is due to Shlyakhtenko (2004).
- Tomatsu-Ueda (2014) proved the "if" part for free product factors and Bernoulli crossed products.
- The general case relies on the type III version of Jones theorem.
- The condition on δ may fail, for example, if M is a full III_1 factor which admits an almost periodic state $\Rightarrow \overline{\delta(\mathbb{R})}$ is compact.

Strongly ergodic equivalence relations

Definition

Let $\mathcal{R}$ be an ergodic non-singular equivalence relation on a probability space (X, μ). We say that $\mathcal{R}$ is *strongly ergodic* if for every sequence $A_n \subset X$, $n \in \mathbb{N}$ such that $\lim_n \mu(\theta(A_n) \triangle A_n) = 0$ for all $\theta \in [\mathcal{R}]$, we have $\lim_n \mu(A_n)(1 - \mu(A_n)) = 0$.

Theorem (Houdayer-M-Verraedt' 2017)

Let $\mathcal{R}$ be a strongly ergodic non-singular equivalence relation on a probability space (X, μ). Then there exist $\theta_1, \dots, \theta_n \in [\mathcal{R}]$ and a constant $\kappa > 0$ such that

$$\forall f \in \mathrm{L}^2(X, \mu), \ \|f - \mu(f)\|_2 \leq \kappa \sum_{k=1}^{n} \|\theta_k(f) - f\|_2.$$

$\Rightarrow$ Application to strong ergodicity of a skew product equivalence relation $\mathcal{R} \rtimes_\Omega G$ where $\Omega : \mathcal{R} \to G$ is a 1-cocycle into a locally compact abelian group G.

Strongly ergodic group actions

Let $\Gamma \curvearrowright (X, \mu)$ be a strongly ergodic action. Then $\mathcal{R}(\Gamma \curvearrowright X)$ is strongly ergodic, hence has spectral gap. But, in general we cannot choose the critical set $\theta_1, \ldots, \theta_n$ inside Γ!

In fact, $\Gamma \curvearrowright (X, \mu)$ need not have spectral gap in general. However, we have:

Theorem (M. 2017)

Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic measure preserving action on a σ-finite space (X, μ). Then the following are equivalent:

- *$\Gamma \curvearrowright (X, \mu)$ is strongly ergodic.*
- *There exists $B \subset X$ with $0 < \mu(B) < +\infty$ such that $\Gamma \curvearrowright (X, \mu)$ has local spectral gap with respect to B.*

Thank you for your attention!